Thermochimica Acta, 207 (1992) 305–312
Elsevier Science Publishers B.V., Amsterdam

Time-domain optimal filtering

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(Received 15 February 1992)

Abstract

A method based on the Wiener approach has been presented for filtering calorimetric signals with the technique of windowing the power spectrum in the frequency domain instead of an abrupt rectangular cut-off of the noise frequencies.

This paper presents a time-domain approach which, for the ideal case of additive noise-corrupted signals with the noise normal, does not require any user input except for considerations of implementation such as filter length. The key point lies in the norming of the final filter coefficients vector for various targets; here the steady state unity gain is selected.

SYMBOLS

$f * g$	f convolved with g
$\theta[f(t)]$	Fourier transform of $f(t)$
$\theta^{-1}[f(t)]$	inverse Fourier transform of $f(t)$
$R_{xx}(\tau)$	autocorrelation function of $x(t)$
P^{-1}	matrix P inverse

INTRODUCTION

Rey [1] has developed the following optimal filter for use with calorimetric signals.

Consider a measured signal $z(k)$ constituted as a signal $x(k)$ corrupted by additive noise $n(k)$

$$z(k) = x(k) + n(k) \tag{1}$$

The problem is to determine a filter $f(k)$ giving $\hat{x}(k)$, the best approximation of $x(k)$ in the least-squares sense

$$\int_{-\infty}^{\infty} |\hat{x}(t) - x(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} |\hat{X}(w) - X(w)|^2 \, \mathrm{d}w \tag{2}$$

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Using the Fourier transforms and

$$\hat{X}(w)=F(w)X(w) \tag{3}$$

Rey's frequency domain solution is

$$F(w)=\frac{|X(w)|^2}{|X(w)|^2+|N(w)|^2} \tag{4}$$

With a cut-off frequency w_c, the optimal rectangular filter is

$$F(w)=\begin{cases}1 & w\le w_c\\ 0 & w>w_c\end{cases} \tag{5}$$

To smooth the resulting parasitic ripples when using the above filter, a window function $W(w)$, (the traditional Welch, Parzen and Hanning windows are studied) is applied to $F(w)$ to construct the more acceptable filter $G(w)$ where

$$G(w)=F(w)\times W(w) \tag{6}$$

Note that, normally, windowing is performed in the time domain

$$g(k)=f(k)\times w(k) \tag{7}$$

where $w(k)$ is the window function, and hence the Fourier transform of eqn. (7) is

$$G(w)=F(w)*\Omega(w) \tag{8}$$

with

$$\theta[w(k)]=\Omega(w)$$

This would have been the form of eqn. (6).

TIME-DOMAIN APPROACH

A time-domain derivation of the filter (eqn. (4)) is given in the Appendix and is

$$A_F(\tau)=\frac{R_{xx}(\tau)}{R_{xx}(\tau)+R_{nn}(\tau)} \tag{9}$$

where

$$A_F(\tau)=\int_{-\infty}^{\infty}f(\tau-u)\,\mathrm{d}u \tag{10}$$

or

$$A_F(\tau)=\frac{1}{1+\mathrm{ISNR}(\tau)} \tag{11}$$

where the function ISNR (which may be seen as an inverse signal-to-noise ratio) is

$$\mathrm{ISNR}(\tau)=\frac{R_{nn}(\tau)}{R_{xx}(\tau)} \tag{12}$$

for a finite impulse response (FIR) filter of length $2M+1$ (two-sided)

$$A_F(\tau)=2\sum_{0}^{M} f_k-\sum_{M-\tau+1}^{M} f_k \tag{13}$$

or

$$A_F(\tau)=\sum_{0}^{M} f_k+\sum_{1}^{M-\tau} f_k \tag{14}$$

Solving for the filter coefficients in eqn. (11) and using eqn. (14) leads to the solution of the simultaneous equation

$$PF=N \tag{15}$$

where the filter coefficients vector

$$F=\begin{bmatrix} f_0 & f_1 & f_2 & \dots & f_{M-1}\end{bmatrix}^T$$

$$N=\begin{bmatrix} n_0 & n_1 & n_2 & \dots & n_{M-1}\end{bmatrix}^T$$

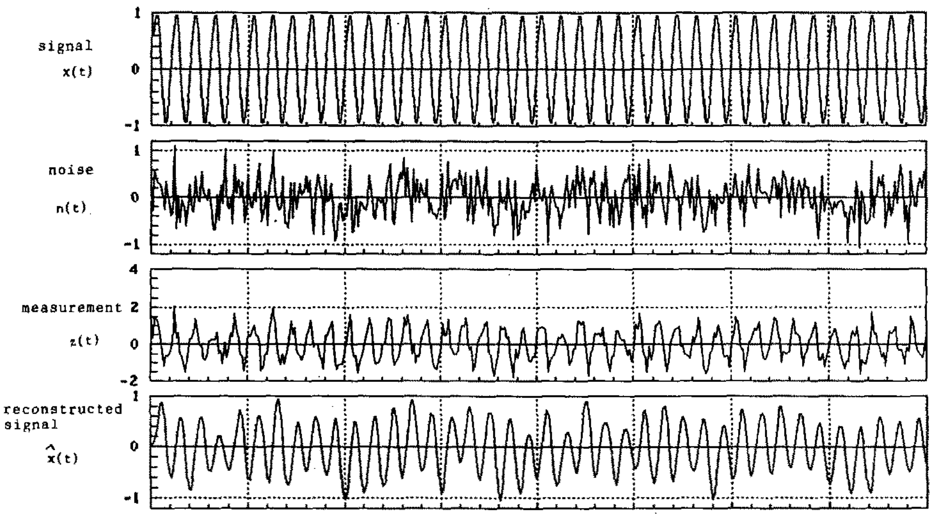

Fig. 1. Reconstruction of sine wave corrupted by white noise of variance 0.16.

with

$$n_j = \frac{1}{1 + \text{ISNR}(j)} \qquad j = 0, \ldots, M-1$$

and P is a square of dimension M and takes the form

$$P = \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 1 & 2 & 2 & \ldots & 2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

The inverse of P is easy to evaluate:

$$P^{-1} = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 & 2 \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & -1 & 0 & \ldots & 0 \\ 1 & -1 & 0 & 0 & \ldots & 0 \end{pmatrix} \qquad (16)$$

and can be used directly to give

$$F = P^{-1} \times N \qquad (17)$$

Convolve the filter with a window function w_k to give new filter coefficients f_k'

$$f_k' = f_f * w_k \qquad (18)$$

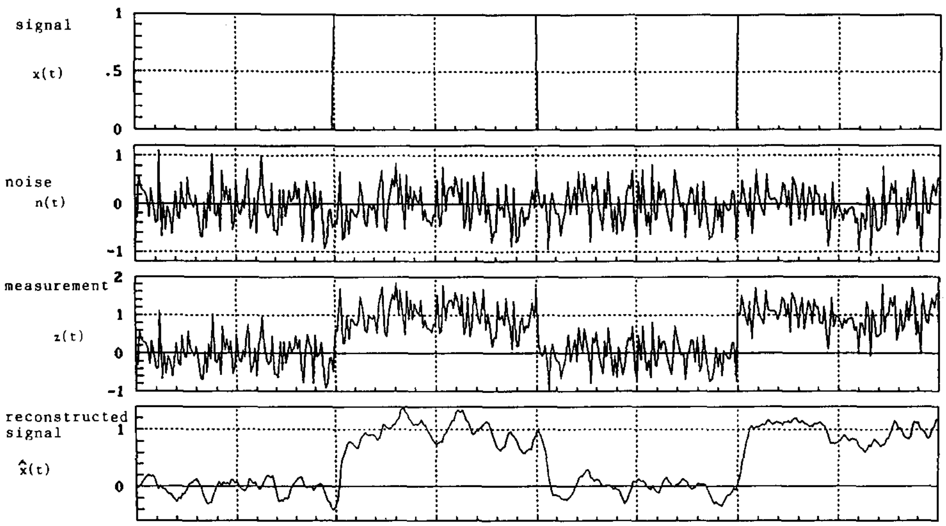

Fig. 2. Reconstruction of pulse wave corrupted by white noise of variance 0.16.

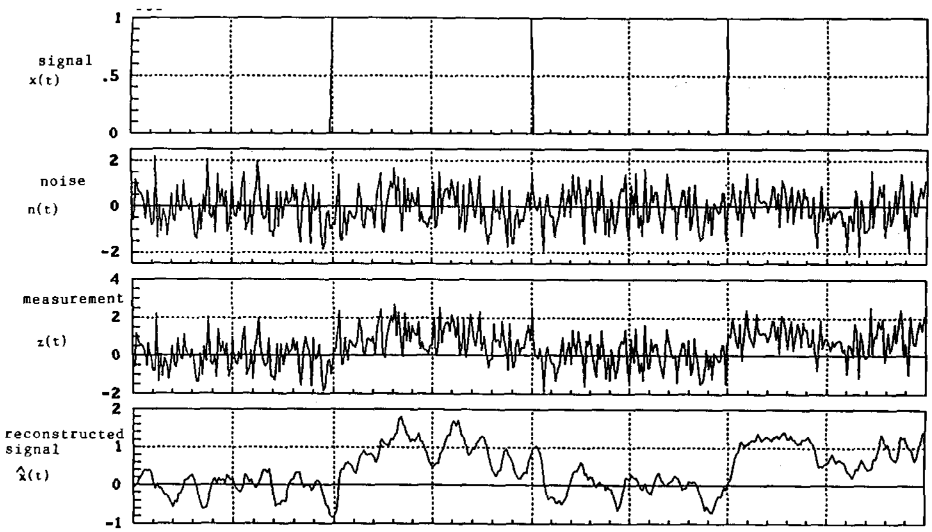

Fig. 3. Reconstruction of pulse wave corrupted by white noise of variance 0.64.

The applied FIR filter F_a is then F' of eqn. (18) normed depending on the final objective

$$F_a = \frac{F'}{\| F \|} \tag{19}$$

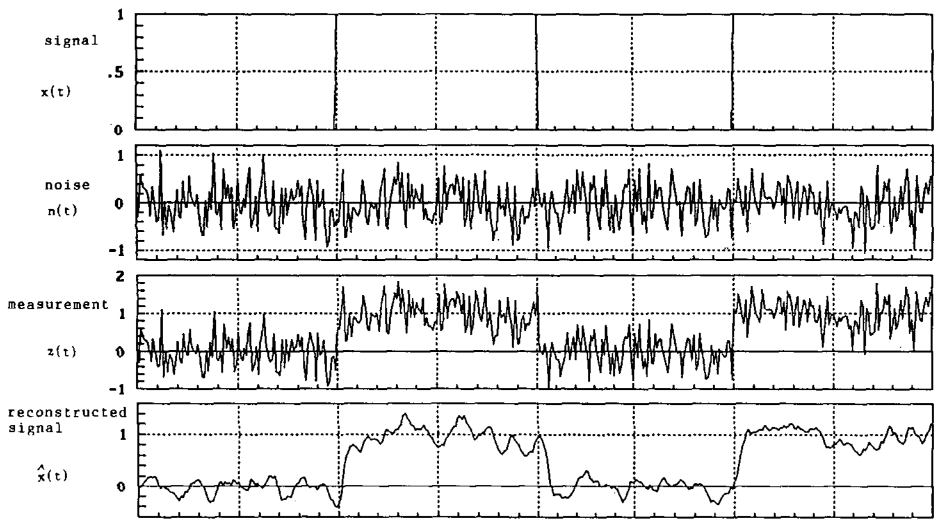

Fig. 4. Reconstruction of pulse wave corrupted by white noise of variance 0.16 (filter calculation: assumes impulse noise autocorrelation).

TABLE 1

Filter simulation parameters

Figure	Signal	White noise variance	Filter length	Filter calculation
1	Sine	0.16	9	Autocorrelation algorithm
2	Pulse	0.16	9	Autocorrelation algorithm
3	Pulse	0.64	9	Autocorrelation algorithm
4	Pulse	0.16	9	Impulse autocorrelation

RESULTS

Tests were performed on various signal types; here, only the sine (Fig. 1) and pulse signals (Figs. 2–4) are presented. The figures show (1) the pure signal, (2) the noise (normal, mean zero), (3) the corrupted measurement, and (4) the reconstruction of the pure signal, i.e. the filtered signal.

The simulation conditions are summarised in Table 1.

The filters in Figs. 1–3 were evaluated by algorithmically evaluating the autocorrelation function of the signal and the noise respectively. For the filter in Fig. 4, the noise autocorrelation was assumed ideal, i.e. impulse, only the zero-shift element of the signal autocorrelation, the r.m.s. value (squared), had to be determined. Unity was selected.

The filter length was determined from the autocorrelation algorithm which gives significant autocorrelation function values only.

Figure 1 shows the effect of the filter on a sine of unit amplitude which is measured with white noise of variance 0.16. The corrupted signal still has a sinusoidal form and the filter restores the amplitude.

The remaining figures illustrate the effect of the filter on pulses with two different noise levels (Figs. 2 and 3) and the use of an ideal noise correlation function with a user input for the signal variance.

CONCLUSION

A simple but effective filter has been presented that requires minimal user tuning. The filter can perform adequately without any user parameter input. Optimal performance is achieved for white-noise corrupted signals. The greatest degradation of performance occurs with signals with high amplitude sinusoidal components.

REFERENCE

1 C. Rey, Optimal filtering of calorimeter signals, Thermochim. Acta, 184 (1991) 329.

APPENDIX

Consider the integrand in eqn. (2) as

$$I = (\hat{x}(t) - x(t))^2 \tag{A1}$$

Differentiating with respect to $f(t)$ for all t (for clarity, the argument is dropped where convenient)

$$\frac{\mathrm{d}I(t)}{\mathrm{d}f(t)} = \frac{\mathrm{d}I}{\mathrm{d}\hat{x}}\frac{\mathrm{d}\hat{x}}{\mathrm{d}f} = 2[\hat{x}(t) - x(t)]\frac{\mathrm{d}\hat{x}}{\mathrm{d}f} \tag{A2}$$

But

$$\hat{x}(t) = \int_{-\infty}^{\infty} f(t-u)z(u)\,\mathrm{d}u \tag{A3}$$

and

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}f} = \int_{-\infty}^{\infty} z(u)\,\mathrm{d}u \tag{A4}$$

Using eqn. (A4) in (A2)

$$\frac{\mathrm{d}I}{\mathrm{d}f} = 2\left[\int_{-\infty}^{\infty} f(t-u)z(u)\,\mathrm{d}u - x(t)\right]\left[\int_{-\infty}^{\infty} z(u)\,\mathrm{d}u\right] \tag{A5}$$

or

$$\frac{\mathrm{d}I}{\mathrm{d}f} = 2\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-u)z(u)z(p)\,\mathrm{d}u\,\mathrm{d}p - \int_{-\infty}^{\infty} x(t)z(u)\,\mathrm{d}u\right\} \tag{A6}$$

Now, the measurement noise is

$$z(u) = x(u) + n(u) \tag{A7}$$

Substituting for $z(u)$ in (A6) gives

$$\frac{\mathrm{d}I}{\mathrm{d}f} = 2\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-u)[x(u)+n(u)][x(p)+n(p)]\,\mathrm{d}u\,\mathrm{d}p \right.$$

$$\left. - \int_{-\infty}^{\infty} x(t)[x(u)+n(u)]\,\mathrm{d}u\right\} \tag{A8}$$

Introducing the variable

$$s = t - u \tag{A9}$$

the second term in eqn. (A8) may be written as

$$\frac{\mathrm{d}I}{\mathrm{d}f} = \int_{-\infty}^{\infty} x(t)x(t-s)\,\mathrm{d}s + \int_{-\infty}^{\infty} x(t)n(t-s)\,\mathrm{d}s = R_{xx}(t) \tag{A10}$$

for $x(t)$ and $n(t)$ uncorrelated.

Now consider the products of the first term of eqn. (A8). The first product is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-u)x(u)x(p)\ \mathrm{d}u\ \mathrm{d}p \tag{A11}$$

Introducing another dummy variable

$$p = t - u \tag{A12}$$

then eqn. (A11) may be written as

$$\int_{-\infty}^{\infty} f(p)\ \mathrm{d}p \int_{-\infty}^{\infty} x(u)x(t-u)\ \mathrm{d}u$$

$$= R_{xx}(t)\int_{-\infty}^{\infty} f(p)\ \mathrm{d}p = R_{xx}(t)A_F(t) \tag{A13}$$

where

$$A_F(t) = \int_{-\infty}^{\infty} f(p)\ \mathrm{d}p \tag{A14}$$

Similarly, the second product is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-u)n(u)x(p)\ \mathrm{d}u\ \mathrm{d}p \tag{A15}$$

$$= \int_{-\infty}^{\infty} f(p)\ \mathrm{d}p \int_{-\infty}^{\infty} n(u)x(t-u)\ \mathrm{d}u = 0 \tag{A16}$$

The third product is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-u)x(u)n(p)\ \mathrm{d}u\ \mathrm{d}p = 0 \tag{A17}$$

for reasons as above.

Finally

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-u)n(u)n(p)\ \mathrm{d}u\ \mathrm{d}p = R_{nn}(t)A_F(t) \tag{A18}$$

Thus for a minimum

$$\frac{\mathrm{d}I(t)}{\mathrm{d}f(t)} = 0$$

and

$$R_{xx}(t)A_F(t) + R_{nn}(t)A_F(t) - R_{xx}(t)$$
$$= A_F(t)[R_{xx}(t) + R_{nn}(t)] - R_{xx}(t) = 0 \tag{A19}$$

from which

$$A_F(t) = \frac{R_{xx}(t)}{R_{xx}(t) + R_{nn}(t)} \tag{A20}$$